HISTORIA MATHEMATICA 21 (1994), 196–203

NOTE

Newton on the Equiangular Spiral: An Addendum to Erlichson's Account

CURTIS WILSON

St. John's College, Annapolis, MD 21404

In a recent article Herman Erlichson called attention to a flaw in Newton's proof of Proposition IX of Book I of the *Principia*. How did Newton fall into this error? A valid proof was near to hand, by an easy addition to Lemma III of Book II; but evidently Newton wished to attempt a different line of argument. The figure for Proposition IX in the first two editions of the *Principia* differs from that in the third edition, and does not involve the quadrilateral "given in kind" that Erlichson rightly objects to. But the basic error remains: the assumption without proof of the similarity of all segments of the spiral with the same central angle. By 1671 Newton had proved this assumption by an integration, establishing the logarithmic property of the equiangular spiral. In Proposition IX it would thus appear that he either simply presented as known an analytic result now familiar to himself, or hoped that his readers would consider it as valid for infinitesimal segments. © 1994 Academic Press, Inc.

Dans un article récent Herman Erlichson a relevé une erreur dans la preuve de la Proposition IX du Livre I des *Principes* de Newton. Comment Newton a-t-il pu tomber dans cette erreur? Une preuve bien fondée était tout près, en se servant d'une addition facilie au Lemme III du Livre II; mais évidemment Newton voulait essayer une ligne différente d'argumentation. La figure pour la Proposition IX dans les deux premières éditions diffère de celle pour la troisième, et ne présente pas le quadrilatère "donné en espèce" auquel Erlichson à juste titre éleve objection. Mais l'erreur fondamentale persiste: la supposition sans preuve de la similitude de tous les segments de la spirale équiangle avec le même angle central. En 1671 ou auparavant Newton avait démontré cette supposition par une intégration, en établissant la propriété logarithmique de la spirale équiangle. Dans la Proposition IX il paraît par conséquent que Newton ou bien avait simplement présenté comme connu un résultat analytique familier à lui-même, ou bien avait espéré que ses lecteurs le tiendraient pour valable dans le cas des segments infinitésimals. © 1994 Academic Press, Inc.

In einem kürzlich erschienenen Artikel lenkt Herman Erlichson die Aufmerksamkeit auf einen Fehler in Newtons Beweis des Lehrsatzes IX im ersten Buch der *Principia*. Wie ist Newton dieser Irrtum unterlaufen? Ein einfacher Zusatz zum Lehnsatz III des zweiten Buches hätte einen zwingenden Beweis geliefert; aber Newton wollte anscheinend eine andere Argumentation durchführen. Die Figur für den Lehnsatz IX in der ersten und zweiten Aufgabe der *Principia* unterscheidet sich von jener in der dritten und umfaßt nicht die vierseitige Figur "gegeben in Spezies," wogegen Erlichson mit Recht Einspruch erhebt. Aber der grundlegende Fehler bleibt: die nicht nachweisbare Annahme der Ähnlichkeit aller Segmente der Spirale mit gleichen zentralen Winkel. Bereits 1671 hatte Newton diese Annahme mittels der Integralrechnung bewiesen und dadurch die logarithmische Eigenschaft der gleichwinkligen Spirale gezeigt. In dem Lehrsatz IX stellt sich jetzt heraus, daß Newton entweder ein ihm selbst vertrautes analytisches Ergebnis als bekannt voraussetzte, oder

0315-0860/94 $6.00

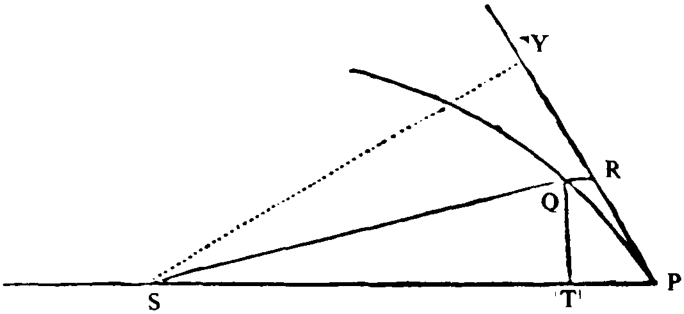

FIG. 1. Newton's diagram for Proposition IX of Book I in the third edition of the *Principia*. (Reprinted by permission of the publisher from *Isaac Newton's Philosophiae Naturalis Principia Mathematica: The Third Edition (1726) with Variant Readings,* edited by Alexandre Koyre and I. Bernard Cohen, Cambridge, MA: Harvard Univ. Press, Copyright © 1972 by the President and Fellows of Harvard College.)

daß er mindestens davon ausging, daß seine Leser es für gültig hielten, wenn die Segmente infinitesimal wären.

AMS 1991 subject classifications: 01A45, 53-03, 53A04.
KEY WORDS: Newton, equiangular spiral, logarithmic spiral.

In a recent article [Erlichson 1992], Herman Erlichson points out a flaw in Newton's proof of Proposition IX of Book I of the *Principia*. Newton in this proposition aims to show that, for a body revolving in an equiangular spiral under the action of a centripetal force tending to the pole of the spiral, the force varies inversely as the cube of the distance from the pole. The spiral is defined merely as "cutting all the radii SP, SQ, &c., in a given angle." As the first step of his proof, Newton claims that, for an "indefinitely small" angle PSQ, the figure SPRQT (see Fig. 1) is "given in kind," because all the angles are given.

But, Erlichson rightly objects, a quadrilateral is not determined by all its angles being given. Newton needed also to show that, for two such quadrilaterals constructed at different places on the same spiral but with the same central angle PSQ = pSq, the corresponding sides are proportional, or SP : Sp = SQ : Sq = QT : qt = QR : qr = PR : pr (see Fig. 2). If Newton's attention had been drawn to this omission, Erlichson proposes, he would likely have had recourse to the circle of curvature, showing first that, for any given equiangular spiral, the radius vector bears a constant ratio to the radius of curvature, and then, by using the circle of curvature to approximate the spiral, showing that all sides of the quadrilateral are proportional to the radius of curvature.

Newton's errors in the *Principia* are not usually elementary. In Book II of the *Principia*, which concerns fluids, "almost all of the results," according to Truesdell, "are original, and but few correct" [Truesdell 1954, p. XII]: the flaws stem

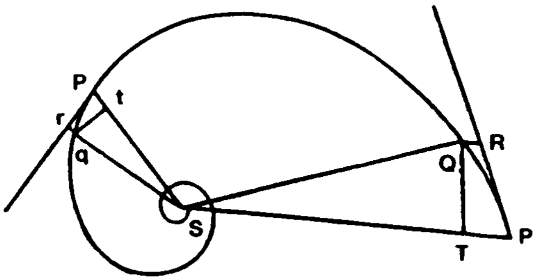

FIG. 2. John Clark's diagram for Proposition IX of Book I in his *A Demonstration of Some of the Principal Sections of Sir Isaac Newton's Principles of Natural Philosophy* (1730).

from the inadequacy of the foundations Newton was able to establish for an almost completely novel inquiry. Something similar must be said of his attempt to derive the precession of the equinoxes in Proposition XXXIX of Book III [Wilson 1987, 238–242]. Is the error in Proposition IX of Book I simply a stupid error, or is there more here that we should try to understand?

We need to know what Newton knew about the equiangular spiral, and when he knew it. First let us take note of the fact that in Lemma III of Book II of the *Principia* Newton proves the property that Erlichson would have him invoke, and goes on to deduce from it a relation from which the result of Prop. IX of Book I easily follows.

Lemma III of Book II is in the service of Prop. XV, which establishes that a body moving in an inverse-square field, where the density of the medium varies inversely with distance from the center of force, and the resistance to motion is assumed to be as the density of the medium and the square of the velocity conjointly, may descend toward the center in an equiangular spiral, provided that the resistance is less than half the centripetal force.

The lemma first establishes that the radius vector in the equiangular spiral bears a constant ratio to the radius of curvature. In the figure for the lemma (here Fig. 3), perpendiculars to the spiral have been drawn at P and Q; they intersect at O. Also, tangents are drawn at P and Q; by the definition of the spiral they make equal angles with their respective radii vectors, SP and SQ. From the right angles OPQ, OQR (Newton here takes the arcs PQ and QR to be identical with the tangents to the spiral at P and Q), let the equal angles SPQ and SQR be subtracted, and there will remain the equal angles OPS, OQS. It follows that a circle passing through the points O, S, P will also pass through Q. Now let Q $\rightarrow$ P, and the circle will become tangent to the spiral at P, and so will cut the line OP perpendicularly; OP becomes a diameter of the circle (it is also equal to the radius of the circle of curvature). The angle OSP, having now come to be inscribed in a semicircle, has become a right angle. It follows that the radius vector, r = SP, is related to the

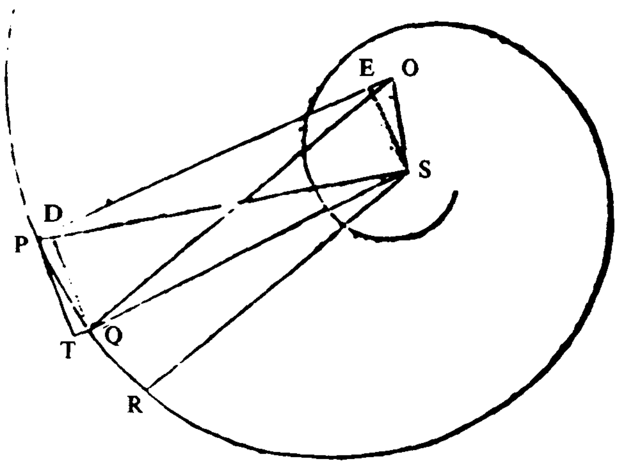

FIG. 3. Newton's diagram for Lemma III of Book II in the third edition of the *Principia* (in the first two editions a single, more complicated diagram serves for both Lemma III and the following Proposition XV). (Reprinted by permission of the publisher from *Isaac Newton's Philosophiae Naturalis Principia Mathematica: The Third Edition (1726) with Variant Readings*, edited by Alexandre Koyre and I. Bernard Cohen, Cambridge, MA: Harvard Univ. Press, Copyright © 1972 by the President and Fellows of Harvard College.)

radius of curvature, ρ = OP, by the relation $r = \rho \sin \alpha$, where α is the angle SPT between tangent and radius vector, and equals the angle SOP.

As a second result Newton establishes the relation

$$\lim_{Q\to P}\left[\frac{PQ^2}{TQ}\right] = 2SP.$$

To show this, he drops QD and SE perpendicular to OP, and thus parallel to PT; the three parallels cut the lines ST and EP into proportional segments, so that

$$\text{TQ} : \text{PD} = \text{ST} : \text{PE}.$$

But as Q → P, ST → SP; and because of the ultimate similarity of the triangles PSE and PSO, SP : PE = ult. 2PO : 2SP. Therefore

$$\text{TQ} : \text{PD} = \text{ult. } 2\text{PO} : 2\text{SP}. \tag{1}$$

As Q → P, PO becomes the radius of curvature, and its double 2PO becomes the diameter of the circle of curvature; hence

$$\text{PD} : \text{PQ} = \text{ult. PQ} : 2\text{PO}. \tag{2}$$

Compounding the ratios on either side of proportions (1) and (2), we obtain

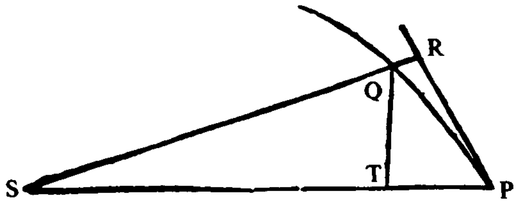

FIG. 4. Newton's diagram for Proposition IX of Book I in the first two editions of the *Principia*. (Reprinted by permission of the publisher from *Isaac Newton's Philosophiae Naturalis Principia Mathematica: The Third Edition (1726) with Variant Readings*, edited by Alexandre Koyre and I. Bernard Cohen, Cambridge, MA: Harvard Univ. Press, Copyright © 1972 by the President and Fellows of Harvard College.)

$$\mathrm{TQ}:\mathrm{PQ} = \text{ult. } \mathrm{PQ}:2\mathrm{SP}, \tag{3}$$

from which it follows that the limiting value of $\mathrm{PQ}^2/\mathrm{TQ}$ is 2SP.

Newton could have used this result to obtain the force law for a body traversing the spiral under the action of a centripetal force directed to S. By the method of Proposition VI, the force is inversely proportional to the limiting value as $\mathrm{Q} \rightarrow \mathrm{P}$ of a quotient: the square of the area SQP swept out by the radius vector, divided by the evanescent subtense (here TQ) of the angle of contact. If we drop from Q a perpendicular Qt on the radius vector SP, the area of SPQ can be expressed (approximately, with ever increasing accuracy as $\mathrm{Q} \rightarrow \mathrm{P}$) by the product $\frac{1}{2}\mathrm{SP} \cdot \mathrm{Qt}$. The force will thus be inversely proportional to $\mathrm{SP}^2 \cdot \mathrm{Qt}^2/\mathrm{TQ}$. Replacing Qt by $\mathrm{PQ} \cdot \sin \alpha$, and $\mathrm{PQ}^2/\mathrm{TQ}$ by its limiting value as previously obtained, namely 2SP, we find that the force is inversely proportional to

$$2\mathrm{SP} \cdot \mathrm{SP}^2 \cdot \sin^2\alpha,$$

and is therefore as $1/\mathrm{SP}^3$.

How did it come about that in Proposition IX of Book I Newton drew a different figure, and so got himself into trouble? The figure for Proposition IX, as Erlichson gives it, was first published in the third edition of the *Principia;* in the first two editions the figure (see Figure 4) was a triangle SPR, together with a line QT, dropped perpendicular to SP from the point Q on the spiral [Koyré & Cohen 1972, I, 113, footnote to the figure between lines 14 and 15]. Newton introduced the revised figure in his annotations on the first edition. Why did he make the change? Probably because Proposition IX depends on Proposition VI, and the figure for Proposition VI shows QR parallel to SP. But whether QR is parallel to SP or in line with SQ makes no essential difference to the applicability of Proposition VI. Proposition VI is based on Lemma XI, and in Case 3 of Lemma XI Newton points out that the lemma is true whether the evanescent subtense of the angle of contact maintains a constant angle to the tangent (as when QR is parallel to SP), or lies in a line verging to a point (e.g., the line SQ).

Of the figure in the first two editions, as of that in the third edition, Newton says that it is "given in kind." Now the triangle SPR (which is identical with the triangle SPT in our Fig. 3) is indeed given in kind, because all its angles are given. Thus for a given central angle θ = PSQ,

$$\mathrm{SR} = \mathrm{SP} \cdot \sin \alpha/\sin(\alpha + \theta), \qquad \mathrm{PR} = \mathrm{SP} \cdot \sin \theta/\sin(\alpha + \theta).$$

Hence the sides SR and PR bear ratios to SP determined by the angle θ. Is the same true for SQ? If so, it could be shown that QR and QT also bear ratios to SP determined by the angle θ. But Q is a point on the spiral, which is defined solely by the condition of cutting all the radii vectores in the same angle α. Newton appears to be assuming the similarity to each other of all portions of the spiral having the same central angle: any portion of it can be magnified or condensed into another portion with the same central angle, by proportional expansion or shrinking of the radii vectors.

Does such a property indeed hold? Yes, and Newton had proved it by 1671, in establishing the logarithmic property of the equiangular spiral.

Already in the mid-1660s, and probably by autumn of 1664, Newton had developed the logarithmic curve as the curve giving the area y under the rectangular hyperbola $z = 1/x$ [Whiteside 1967, 376–377, n. 47]; in modern terms his result is

$$y = \int_1^x \frac{1}{x} \cdot dx = \log x.$$

Presumably he was not initially aware of the logarithmic character of the curve. By late 1664 he had also introduced the idea of continuously compounded interest, which leads to the same mathematical problem [Whiteside 1967, 461–462].

It is in the text often referred to as *Methods of Series and Fluxions,* completed in 1670–1671, that Newton explicitly announced the logarithmic property of the equiangular spiral. Here he developed a general formula for the radial component of the radius of curvature of a curve given in polar coordinates [Whiteside 1969, 169–173]:

$$\rho \sin \alpha = \frac{r(1 + z^2)}{(1 + z^2 - dz/d\theta)},$$

where ρ is the radius of curvature, r is the radius vector, given as a function of the central angle θ, α is the angle between radius vector and tangent, and z is $(1/r)dr/d\theta$, and is equal to cot α. In the particular case of the equiangular spiral, α and hence cot α are constants, and $dz/d\theta$ is therefore zero. It follows from the above formula that $\rho \cdot \sin \alpha = r$, the same result that was obtained in Lemma III of Book II of the *Principia*.

But Newton went further in his analysis. If

$$(1/r)dr/d\theta = \cot \alpha,$$

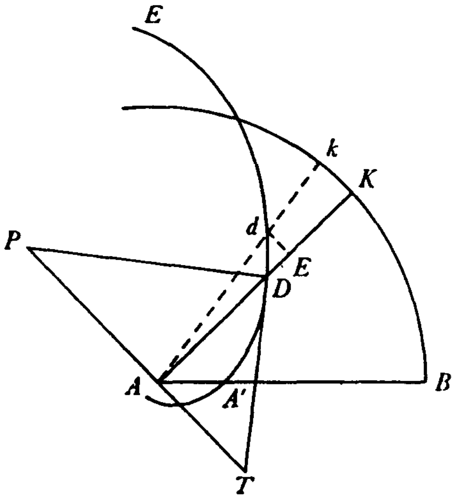

FIG. 5. Newton's diagram for the logarithmic spiral in his *Methods of Series and Fluxions* (From *Isaac Newton's Mathematical Papers*, Vol. III (1969), edited by D. T. Whiteside. Reprinted with permission of Cambridge University Press.)

where α is a constant, an integration can be performed that is formally similar to the integration in rectangular coordinates giving the area under the equilateral hyperbola. It follows that

$$\log r_2 - \log r_1 = (\theta_2 - \theta_1) \cot \alpha,$$

or

$$\log (r_2/r_1) = (\theta_2 - \theta_1) \cot \alpha,$$

showing that the ratio of the two radii vectors r_2 and r_1 depends solely on the central angle between them. Newton must have derived these results, for he writes [Whiteside 1969, 186–187]:

> By the 'quality' of curvature I understand its form in so far as it is more or less inequable, in other words, as it varies more or less in proceeding along different parts of the curve. So to anyone enquiring what the quality of curvature of a circle is, the reply can be given that it is uniform or invariant; while to any one asking the quality of curvature of a spiral described [see Fig. 5] by the motion of the point D proceeding from A with accelerated speed AD in the straight line AK as it rotates uniformly around the centre A (so that the line AD bears to the arc BK described by the given point K the ratio of a number to its logarithm), the answer can be made that it is uniformly variant or equably inequable. . . .
>
> 1. At points similarly located in similar curves there is a like inequability or variation of curvature.
>
> 2. The moments of the radii of curvature at those points are proportional to the contemporaneous moments of the curves, and their fluxions correspondingly so. . . .

Newton thus sees the equiangular spiral as having the "quality" of curvature next simplest after the circle: a curvature that is "equably inequable," or such that the rate of increase or decrease in the radius of curvature with angle is proportional to the radius of curvature, and consequently the rate of change of the radius vector with angle proportional to the radius vector.

I conjecture that when Newton looked at Fig. 4, he saw the figure SPRQT as "given in kind" because of his prior knowledge about the spiral—its "self-similarity" in all its parts. Perhaps he attempted to reason silently with an imagined reader: Can't you see that if the tangent always makes the same angle with the radius vector, the decrease or increase in the radius vector ($dr = r \cdot \cot \alpha \cdot d\theta$) will vary as the radius vector (r) itself, so that whatever the size of SP, SQ for a given angle, PSQ, must bear to SP a fixed ratio? Or perhaps—for he was often hard on his readers—he merely took as known a result now familiar to himself. In either case, it would appear that Newton's attempted proof involves an inadequate translation into geometry of a proof earlier achieved analytically.

ACKNOWLEDGMENTS

I am indebted to Professor Erlichson for bringing home to me the flaw in Newton's proof of Proposition IX. For the reference to Newton's analytic derivation of the logarithmic property of the equiangular spiral, I am indebted to Michael Nauenberg, "Newton's Early Computational Method for Dynamics," *Archive for History of Exact Sciences,* forthcoming.

REFERENCES

Erlichson, H. 1992. Newton's solution of the equiangular spiral problem and a new solution using only the equiangular property. *Historia Mathematica* **19,** 402–413.

Koyré, A., & Cohen, I. B. 1972. *Isaac Newton's Philosophiae Naturalis Principia Mathematica: The Third Edition with Variant Readings.* Cambridge, MA: Harvard Univ. Press.

Truesdell, C. A. 1954. Rational fluid mechanics, 1687–1765. In *Leonhardi Euleri Opera Omnia,* II, Vol. 12.

Whiteside, D. T. 1967. *The Mathematical Papers of Isaac Newton,* I, 1664–1666. London, New York: Cambridge Univ. Press.

Whiteside, D. T. 1969. *The Mathematical Papers of Isaac Newton,* III, 1670–1673. London/New York: Cambridge Univ. Press.

Wilson, Curtis, 1987. D'Alembert *versus* Euler on the precession of the equinoxes and the mechanics of rigid bodies. *Archive for History of Exact Sciences* **37,** 233–274.